Letters

Comments on "Internal Impedance of Conductors of Rectangular Cross Section"

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In the above paper,[1] Antonini *et al.* discuss some interesting questions concerning the ac internal impedance of rectangular cross-sectional conductors (such as those typically used as transmission lines and interconnects in many microwave and digital printed circuits). The authors of the above paper present calculations showing that the real (R) and the imaginary (ωL_i) parts of the alternate current (ac) internal impedance ($Z = R + j\omega L_i$) are not identical for that type of conductors even if a strong skin-effect condition prevails. Following the authors discussion, this would be in sharp contrast with the equality ($R = \omega L_i$) commonly accepted in textbooks for very high operation frequency. If $R = \omega L_i$ can be assumed, the internal inductance L_i can be trivially obtained from the calculated value of R. However, this assumption is what is considered wrong or inaccurate in the above paper for the specific case of rectangular cross-sectional conductors. On the other hand, the authors of the above paper raise a question about the foundations of the well-known Wheeler's rule when applied to rectangular conductors. Apparently Wheeler's rule relies on the high-frequency resistance and internal inductive reactance being equal. However, although this premise seems to be violated in the case of rectangular conductors, Wheeler's rule yields reasonably accurate results in this case. We would like to share with the authors of the above paper and other interested colleagues some considerations concerning the above-mentioned conclusions and paradox.

From the reading of section III in the above paper, it seems that the authors attribute the difference between the real (R) and imaginary (ωL_i) parts of the internal impedance to the existence of right-angle corners in the rectangular cross-sectional geometries. These corners would be responsible for the nonconstant distribution of the currents along the periphery of the conductors, and this fact makes it a different internal resistance and reactance. However, in our opinion, there is a problem hidden in this argument. In order to illustrate the nature of the problem, let us compare some numerically computed results reported in the above paper with results analytically obtained for two simpler geometry wires having the same dc resistance ($R_{\rm dc}$). The cross sections of the structures under comparison are shown in Fig. 1, where conductivity and dimensions are given. Note that the surface of the cross sections and the conductivity are the same for all the three wires, thus, they have all the same value of $R_{\rm dc}$. Fig. 1(b) is an idealized version of the square wire considered in the above paper [see Fig. 1(a)], where corner effects are suppressed by imposing magnetic boundary walls at two parallel sides. Closed-form expressions for the internal impedance of this structure are readily obtained through elementary calculations. These expressions are valid for any frequency, including

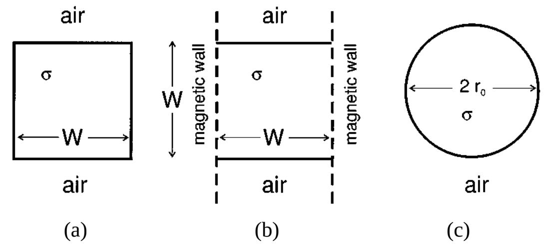

Fig. 1. Cross section of the wires whose internal impedances are compared in Fig. 2. (a) An example structure analyzed in Fig. 6 of the above paper. (b) The same structure with magnetic walls eliminating corner effects. (c) Round wire having the same dc resistance as the previous wires. Dimensions: $W = 4.62$ mm, $r_0 = W/\sqrt{\pi}$. Conductivity: $\sigma = 5.72 \times 10^7$ $(\Omega\cdot \text{m})^{-1}$.

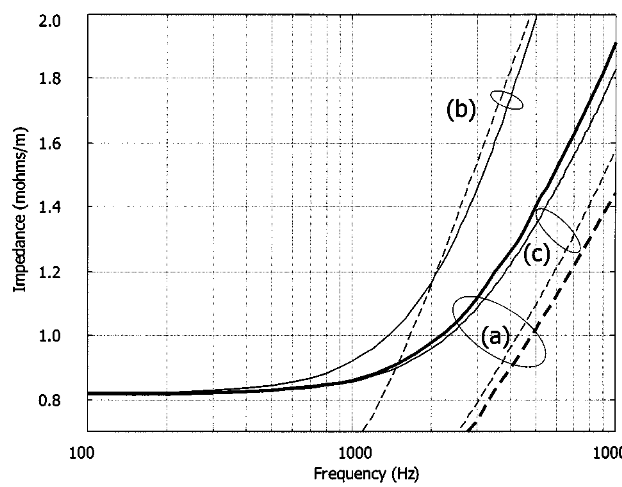

Fig. 2. Internal resistances (solid lines) and reactances (dashed lines) of the wires in Fig. 1. Thick lines correspond to data appearing in Fig. 6 in the above paper for the structure in Fig. 1(a).

negligible to strong skin-effect operation. Fig. 1(c) is a cylindrical circular wire whose internal impedance is known in closed form in terms of Bessel functions for any frequency (see, e.g., [1, pp. 180–186]). We have reproduced in Fig. 2 the numerically computed data reported in Fig. 6 in the above paper [internal resistance and reactance of the structure in Fig. 1(a)] and analytical exact data for the structures in Fig. 1(b) and (c). We can see that the real and imaginary parts of the internal impedance are very close in the case of Fig. 1(b). There is a small difference because, for the range of frequencies included in this figure, the skin depth is not yet negligible in comparison with W. However, it is true that if frequency is increased, R becomes identical to ωL_i in this case. However, the internal impedance of the circular cross-sectional wire [see Fig. 1(c)] presents a behavior very similar to the one of the square cross-sectional wire, even though the round wire obviously has no corners. In our opinion, for the structure shown in Fig. 1(b), we find $R = \omega L_i$ (when skin effect is developed) because the electromagnetic field (EMF) inside the conductor is exactly a uniform plane wave, as assumed in the derivation of the high-frequency internal impedance formulas, and not only because the EMF is constant along the periphery of the conductor. In fact, the EMF is uniform around the circular wire surface, but $R \neq \omega L_i$ because a cylindrical wave instead of a plane wave exists inside the conductor. The cylindrical solution approaches

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the plane wave limit only if $\delta \ll r_0$ (δ is the skin depth). This result can be generalized saying that, in order to have $R \approx \omega L_i$, the curvature radius must be large in comparison with the skin depth. In the case of rectangular cross-sectional conductors, this condition is not fulfilled in the vicinity of the corners. Thus, even though each side is a planar surface, the EMF is strongly nonuniform along the conductor surface (as it is stated in the above paper), thus providing a place to an inner EMF that is not the one assumed in the derivation of the conventional strong skin-effect formulas. There is still a noteworthy point to be emphasized in connection with this question. From inspection of Fig. 2 and from analytical calculations, we can verify that, if the skin effect is strong, $R \approx \omega L_i$ in the sense $(R - \omega L_i) \rightarrow 0$ when $\omega \rightarrow \infty$, only in the case of the structure in Fig. 1(b). In the case of the round wire [see the structure in Fig. 1(c)], it can be readily demonstrated that $R - \omega L_i \rightarrow 0.25\ R_{\rm dc}$ when $\omega \rightarrow \infty$ (this difference between R and ωL_i can be appreciated, e.g., in [1, Fig. 4.5(a) and (b)]). This difference is negligible for very high frequencies because, in such case, $R \approx \omega L_i \gg R_{\rm dc}$. Therefore, we can always say that $R/(\omega L_i) \rightarrow 1$ when $\omega \rightarrow \infty$, and it is in this sense that Wheeler formulas are correct. However, the above-mentioned difference becomes important when the cross-sectional dimensions are just a few times the skin depth, as happens in Fig. 2. This is the difference that is observed in the curves shown in the above paper. As can be seen, this difference cannot be considered a specific feature of rectangular cross-sectional conductors or exclusively related with the presence of corners. Note that, if we are correct, all this would provide an answer to the open question formulated in the last sentence of the above paper, namely, "The reason why Wheeler's rule seems to give reasonably accurate values of resistance when skin effect is well developed even though the basic premise is violated is not understood." The problem is in the concept "well developed skin effect." The authors of the above paper seem to consider that the skin effect is well developed if the transverse dimensions of the wire are a few times the skin depth. However, from the study of the circular wire, it is obvious that the resistance coincide with the "high frequency" (Wheeler) formulas when skin depth is meaningfully small in comparison with the curvature radius of the conductor contour. If this condition is not fulfilled, Wheeler's expressions would be just approximated, independently of the existence of corners. However, it is true that the existence of corners makes the difference between R and ωL_i larger than if corners were not present, but this is a rather obvious result.

There is still another point deserving attention in connection with conductors having right angles such as those involved in this paper. Let us consider the square conductor of the example in the previous paragraph as one of the conductors of a two-wire transmission line. The total per unit length (p.u.l.) inductance of the transmission line, if strong skin-effect operation is assumed, can be usually split into two contributions: the external inductance, associated to the magnetic field existing in the dielectric medium around the wire, and the internal inductance, associated to the magnetic field penetrating the conductor. The external inductance is considered to be identical to the inductance of the transmission line when the conductors are considered perfect. This assumption is justified at any frequency (including weak skin-effect operation) for structures having a high degree of symmetry. For instance, the external inductance of a conventional coaxial transmission line is the same under weak or strong skin-effect conditions: the external magnetic field does not depend on the particular distribution of the current inside the inner conductor, whereas cylindrical symmetry is respected. For more general situations, the assumption of an external magnetic field identical for the line made of perfect and nonperfect conductors is valid if the skin depth is very small in comparison with local curvature radius of the contour curve describing the geometry of the conductor. It is clear than for rectangular cross-sectional wires, the external magnetic field is not identical to the field for the structure with perfect conductors, at least in the neighborhood of the corners. Therefore, we cannot rigorously say that the total inductance is the summation of the internal inductance and the inductance of the same structure made of perfect conductors. A more general interpretation of the model would be based on the consideration of an *incremental* inductance associated to the nonperfect nature of the conductors rather than an *internal* inductance. This incremental inductance would be the difference between the total inductance of the lossy line and the total inductance of the lossless line. This parameter would not coincide with the internal inductance computed from the internal magnetic energy stored inside the conductors, such as it is carried out in the above paper. The external inductance could be calculated from the computed values of the current density and magnetic potential vector inside the lossy conductors since the expression of the magnetic energy in terms of such quantities includes the true external magnetic energy. Now, by substracting the inductance of the lossless transmission line, we have the incremental inductance, which should be used to model the inductive effect due to the lossy nature of the conductor instead of the internal inductance. Obviously, for very strong skin-effect operation or for conductors having smooth contours, the distinction between incremental and internal inductance has no meaning. It would be interesting to explore which of those inductances (incremental or internal) yields a reactance closer to the ac resistance in the case of rectangular conductors or conductors having corners.

References

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Authors' Reply

Giulio Antonini, Antonio Orlandi, and Clayton R. Paul

The authors of the above paper,[1] intended to demonstrate the following point about conductors that have rectangular cross sections. The per-unit-length resistance r and per-unit-length internal inductive reactance ωl_i are considerably different at high frequencies where skin effect is well developed, which is in contrast to conductors of circular–cylindrical cross section (wires). There is the question of what is meant by skin effect being "well developed." First, consider the case of a wire of radius r_w. We compare that radius to a skin depth $\delta = 1/\sqrt{\pi f \mu \sigma}$, where f is the frequency of excitation,

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TABLE I
RATIO OF RESISTANCE TO INTERNAL INDUCTIVE REACTANCE VERSUS THE CONDUCTOR HALF-WIDTH IN SKIN DEPTHS FOR A 4.62 mm × 4.62 mm CONDUCTOR

Frequency	$\frac{r}{\omega l_i}$	Conductor half width (2.31mm) in skin depths
100 MHz	1.314	350δ
500 MHz	1.248	782δ

$\mu = 4\pi \times 10^{-7}$ is the permeability, and $\sigma = 5.8 \times 10^7$ is the conductivity of the metal that is assumed to be copper in the above paper. For frequencies where $r_w/\delta > 1$, the resistance begins to increase from its dc value as $\sqrt{f}$ and the internal inductance decreases at that rate. It is demonstrated in [1, p. 182] that the resistance and internal inductive reactance of a wire converge as the frequency increases without bound, i.e., $r/\omega l_i \rightarrow 1$ as $\omega \rightarrow \infty$. The question here is how the rate of convergence for wires compares to the rate of convergence for conductors having rectangular cross sections and 90° corners. This behavior is graphed for wires in [1, Figs. 4.5a and b]. The convergence is rather rapid initially once the wire radius exceeds the skin depth. For the highest frequency shown in [1, Fig. 4.5b], where the wire radius is 14 skin depths, i.e., $r_w/\delta = 14$, the ratio of resistance to internal inductive reactance is about $r/\omega l_i \cong 1.05$, and in the range $r_w/\delta > 4$, the convergence appears to proceed rather slowly. Based on this rate of convergence, we might say that skin effect seems to be rather well developed with respect to this rate of convergence when the wire radius is on the order of 4–6 skin depths.

Now contrast this with the results for conductors of rectangular cross section that were investigated in the above paper. Several cross sections were investigated, i.e., 1.4 mil × 15 mil, 50 μm × 50 μm , and 4.62 mm × 4.62 mm. Observe the case of the 4.62 mm × 4.62 mm cross section, where the resistance and internal inductive reactance are given at 100 MHz in Table I of the above paper as $r/\omega l_i \cong 1.314$. In other words, the resistance is some 31% higher than the internal inductive reactance at 100 MHz. However, at 100 MHz, one-half of the width (2.31 mm) is 350 skin depths. These results have been recently recalculated for a frequency of 500 MHz. The ratio of resistance to internal inductive reactance is $r/\omega l_i \cong 1.248$. In other words, the resistance is some 25% higher than the internal inductive reactance at this higher frequency of 500 MHz. However, at 500 MHz, one-half of the width (2.31 mm) is 782 skin depths. These results are tabulated in Table I. This indicates that the convergence of resistance to internal inductive reactance for rectangular cross sections can be much slower than for the case of a wire. Unlike the case of a wire, which can be solved analytically, the rectangular cross-section case has not been solved analytically. However, like the wire, it seems plausible to expect the resistance and internal inductive reactance to converge as frequency increases without bound. However, our data indicate that this convergence occurs much more slowly than for a wire.

In the above paper, we commented that Wheeler's rule is frequently used to compute high-frequency resistance and loss for conductors of rectangular cross section such as a microstrip. However, Wheeler's rule ideally requires that the high-frequency resistance and internal inductive reactance be equal in order to be valid. Wheeler also pointed out that his rule is valid only for conductors whose radius of curvature is much greater than the skin depth. While these restriction can be reasonably satisfied for wires, they do not seem to be as well satisfied for rectangular cross sections as for wires. Regarding the accuracy of Wheeler's rule when used for conductors of rectangular cross section, consider Holloway and Kuester's paper [2], in which they compare the results of Pucel *et al.* [3], where the losses are computed using Wheeler's incremental inductance rule, to a new formulation and methods of computation. They show for microstrip lines that results such as in [3], which are based on Wheeler's rule, can give loss predictions that are in error by some 12%–30%. The authors may wish to consider other papers by Holloway and Kuester regarding the effect of edge shape on conductor loss, e.g., [4] and [5]. These papers tend to support the notion that when the ratio of thickness of rectangular conductors to skin depth becomes large, edge shape can be important and may become important in calculating conductor loss unlike wires. The authors state in their comments, "However, it is true that the existence of corners makes this difference between R and ωL_i larger than if corners were not present, but this is a rather obvious result." On the contrary, it appears through the extensive use of results based on Wheeler's rule to compute loss that this is not so obvious.

Finally, the authors comment on the case of two conductors and the separation of the total per-unit-length inductance of two-conductor lines into internal and external inductance components. While their suggestions may have merit, we only intended to address the single-conductor case in the above paper.

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